Continuity Project

By: Abduljaleel Al-Saleh & Yousef Jeragh
Class: 12B

What is the concept of continuity and what are its 3 conditions?

Definition: continuous at a point

A function $f(x)$ is **continuous at a point a** if and only if the following three conditions are satisfied:

i. $f(a)$ is defined

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point a** if it fails to be continuous at a.

Examples Using the 3 Conditions

Discontinuous

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$$

$\boxed{1}$ at $x=2$ $f(2)$ is undefined (No equal sign)

$\boxed{a}$ $\Rightarrow$ f is not continuous at $x=2$

$\boxed{b}$ $\lim_{x\to 2^-} 3 - x = -2 = 1$

$\lim_{x\to 2^+} \frac{x}{2} + 1 = \frac{2}{2} + 1 = 2$ $\neq$ $\lim_{x\to 2} f(x)$ dne $\Rightarrow$ disc is NR

$\boxed{c}$ Classify: jump disc ($\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$ + no infinity)

DISCONTINUOS

Continuous

Example 1:

If $f(x) = \begin{cases} 3 + 2x & x < 1 \\ 6 - e^{(x-1)} & x = 1 \\ 2x^3 - x + 4 & x > 1 \end{cases}$

1) Is f(x) continuous at x=1 ? Justify

$\boxed{1}$ $f(1) = 6 - e^{1-1} = 6 - e^0 = 5$ ✓

$\boxed{2}$ $\lim_{x\to 1^-} 3 + 2x = 3 + 2 = 5$

$\lim_{x\to 1^+} 2x^3 - x + 4 = 2 - 1 + 4 = 5$ $\lim_{x\to 1} f(x) = 5$ ✓

$\boxed{3}$ $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = 1$.

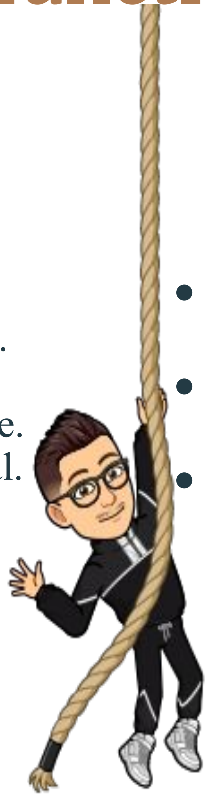
CONTINUOS

How do you know if a function is continuous or discontinuous?

Algebraically

- Find if there is an image from the piecewise functions(equal to or equal).
- Evaluate that there is a limit from the left and right of a certain point in a line.
- See if the image and the limit are equal.

If all are satisfied, then the function is continuous. If at least one is not satisfied, then the function is discontinuous.

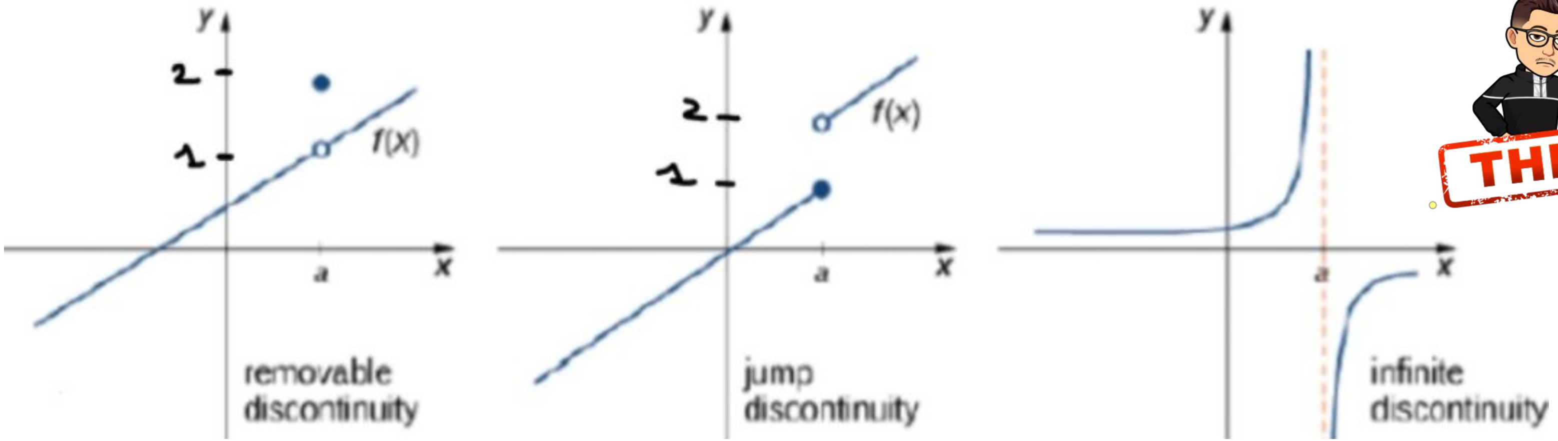


Graphically

- Find if there is an image on the graph (a shaded point).
- Identify the limit for the point that has the same x coordinates as the image.
- See if the image and the limit are equal.

If all are satisfied, then the function is continuous. If at least one is not satisfied, then the function is discontinuous.

The 3 Types of Discontinuity-Graphically



The third condition is not satisfied

The second and third conditions are not satisfied

All of the conditions are not satisfied

Removable Discontinuity (Algebraically)

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ x/2, & x > 2 \end{cases}$$

$\boxed{a}$ at $x = 2$

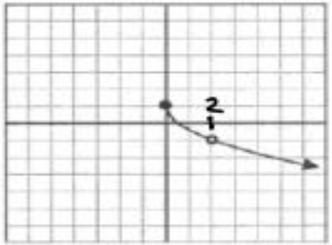
- $f(2) = 2$ ✓
- $\lim_{x \to 2^-} 3 - x = 3 - 2 = 1 = \lim_{x \to 2} f(x) = 1$

  $\lim_{x \to 2^+} \frac{x}{2} = \frac{2}{2} = 1$
- $\lim_{x \to 2} f(x) = 1 \neq f(2) = 2$

$\boxed{b}$ yes it is removal (because $\lim_{x \to 2} f(x)$ exists) f is not continuous (fails to satisfy the 3rd cond)

$\boxed{c}$ $f(2) = 1$ so f will be continuous at $x = 2$ (to remove the dis the $f(a)$ has to be equal to $\lim f(x)$)

Removable discontinuity (Graphically)



(a) at $x=2$

(b) $f(2)$ is undefined (1st cond)

(c) removable disc

(d) R / $f(2)=-1$

because $\lim_{x\to 2} f(x) = -1$

Jump Discontinuity (Algebraically)

Is $f(x)$ continuous at $x = 3$ if $f(x) = \begin{cases} x + x^2 & x < 3 \\ 3 + 3x & x = 3 \\ 4x + 1 - ln(2x - 5) & x > 3 \end{cases}$

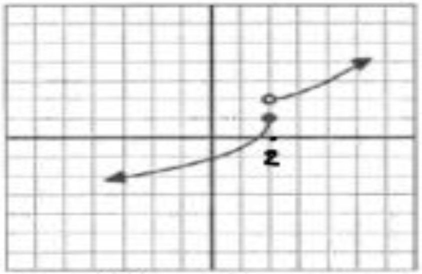
① $f(3) = 3 + 3(3) = 12$ ✓

② $\lim_{x \to 3^-} x + x^2 = (3) + (3)^2$
$= 3 + 9 = 12$

③ $\lim_{x \to 3^+} 4x + 1 - \ln(2x - 5)$
$= 4(3) + 1 - \ln(6 - 5)$
$= 13$

$\Rightarrow f$ is discontinuous at $x = 3$
$\lim_{x \to 3} f(x)$ DNE (2nd cond fails to be satisf)

Jump discontinuity (Graphically)



(a) at $x=2$

(b) $\lim_{x\to 2} f(x)$ dne (2nd cond)

(c) jump disc

(d) NR | $\lim_{x\to 2} f(x)$ dne

Infinite Discontinuity (Algebraically)

$$f(x) = \begin{cases} \dfrac{1}{x-1}, & x < 1 \\ x^3 - 2x + 5, & x \ge 1 \end{cases}$$

(a) at $x=1$

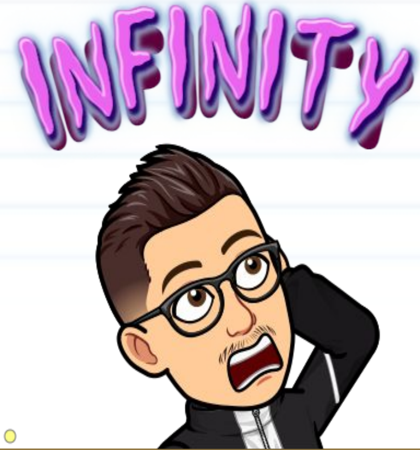
① $f(1) = (1)^3 - 2(1) + 5$

$= 1 - 2 + 5 = 4$ ✓

② $\lim\limits_{x \to 1^-} \dfrac{1}{x-1} = \dfrac{1}{1-1} = \dfrac{1}{0} = \infty$

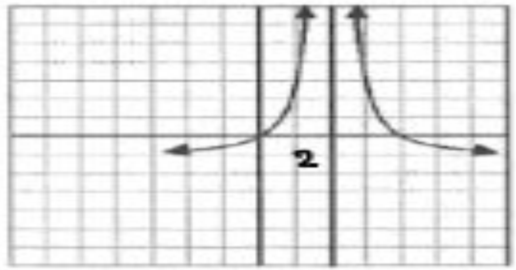
$\nexists \lim\limits_{x \to 1} f(x)$ dne

[b] infinite discontinuity (c) NR ($\lim\limits_{x \to 1} f(x)$ dne)



Infinite Discontinuity (Graphically)

4)



ⓐ at $x = 2$

ⓑ $f(a)$ is undefined (1st cond)

ⓒ infinite dis

ⓓ NR) $\lim_{x \to 2} f(x)$ dne

Extended Function Example

EXTENDED

Extended Functions can be produced when the original function is a removable discontinuity. We can remove a discontinuity when the function fails to satisfy the third condition but its limit exists.

$$f(x) = \frac{x^2 - 9}{x + 3}$$

1st domain: $x + 3 = 0$

$x = -3$

$\mathbb{R} - \{-3\}$

$9 - 9 = 0$

2nd step: $\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = \frac{0}{0} = \lim_{x \to -3} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = \lim_{x \to -3} x - 3 = -3 - 3 = -6$ ✓

$-3 + 3 = 0$

(Extended function can be built only for points where we have R disc)

3rd step: $g(x) = \begin{cases} \frac{x^2 - 9}{x + 3} & \text{if } x \neq -3 \\ -6 & \text{if } x = -3 \end{cases}$

When can we talk about removable discontinuity?

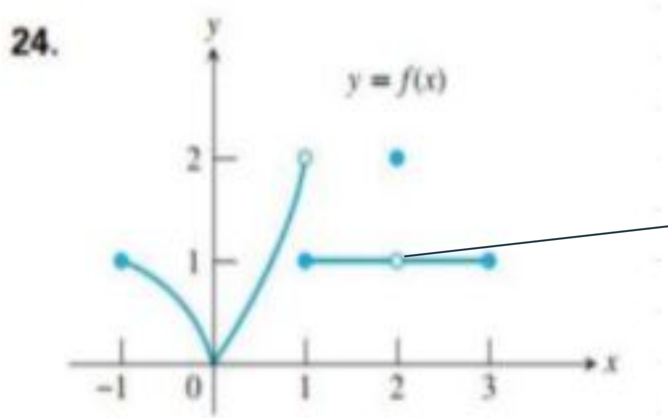
We can talk about about a removable discontinuity when a limit exists and when the third condition is dissatisfied. However, whether the first condition is satisfied or not it will not prevent the discontinuity from being removable, as long as the third condition is dissatisfied.

Can a function be continuous at a point but does not have limit?

No it can not be continuous because then it will fail to satisfy the second condition and as we know if even one of the three conditions are not satisfied then a function can not be continuous.

Can a function have a limit at a point and be not continuous?

Yes. When it does not have an image (first condition) or if the image and the limit are not equal (third condition)

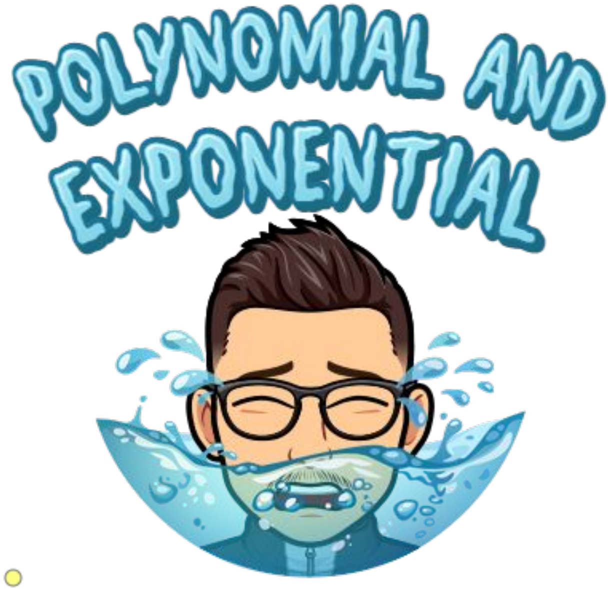


At x=2, there is a limit, but it is discontinuous because the image(2) does not equal the limit(1).

What kind of functions is continuous for all real numbers?

Polynomials: (2x^3 - 5x^2 +10x + 20)
All polynomials have no restrictions on their domains

Exponential Functions: a^x (a being any number)



Two functions that are continuous on their domain:

$f(x) = x^3 - x^2 + 1 \; ; [1,2]$

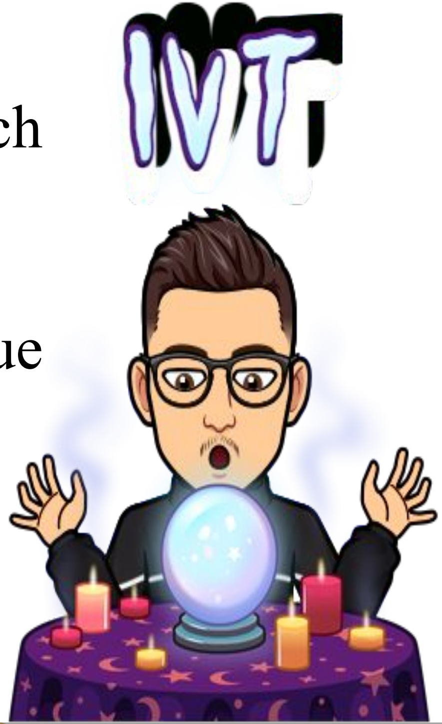
f is polynomial so continuous everywhere so continuous on $[1,2]$

$f(x) = e^{-x} + 2 \; ; [0,3]$

e^{-x} is an exponential function continuous everywhere
2 is constant is continuous everywhere

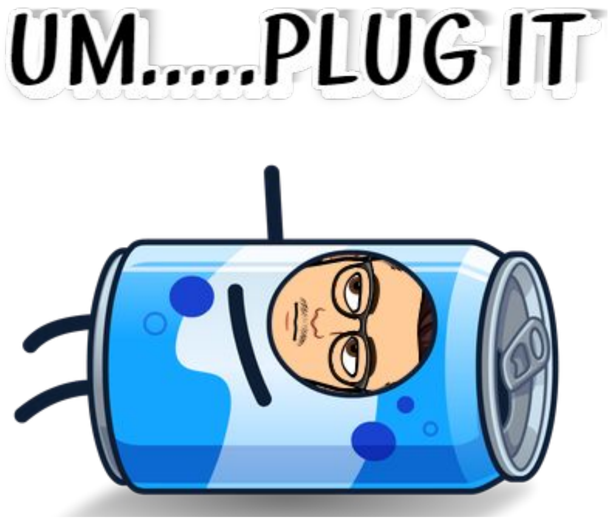
What is the intermediate value theorem?

The intermediate value theorem is a theorem which states that if f(x) is a continuous function whose domain contains the interval [a, b], then at some point within the interval it takes on any given value between f(a) and f(b).



How can we confirm that we are able to apply the IVT?

When we plug the two given x points into the function, the answers should be one negative and another one positive.
If both the answers are negative or both are positive, then the IVT cannot be applied.



Example where can we apply the IVT

$f(x) = x^3 + x - 5$; $[\overset{a}{1}, \overset{b}{2}]$

• polynomial function is continuous everywhere

$-\infty \quad [1, 2] \quad \infty$

① • f is a polynomial function continuous everywhere so continuous over $[1,2]$

② $f(1) = (1)^3 + 1 - 5 = -3$

$f(2) = 2^3 + 2 - 5 = 8 + 2 - 5 = 5$

$[-3, 5]$

③ $0 \in [-3, 5]$

$\Rightarrow$ using the ivt $f(x) = x^3 + x - 5$ has at least one $c \in [1,2]$ such as $f(c) = 0$

Example where we can't apply the IVT

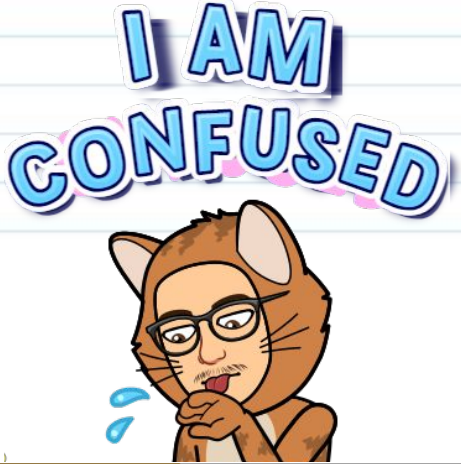
$f(x) = x^3 - x^2 + 1$; $[1,2]$

① • f is polynomial so continuous everywhere so continuous on $[1,2]$

② $f(1) = (1)^3 - (1)^2 + 1 = 1$

$f(2) = (2)^3 - (2)^2 + 1 = 5$

③ $0 \notin [1,5]$ we can't use the ivt



Thank You For Your Attention